Solitons reflection in boundary value problems and a nonlinear method of images

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Abstract:

One of the hallmarks of integrability is the existence of exact N-soliton solutions. It is well known that each soliton is associated to a discrete eigenvalue for the scattering problem via the inverse scattering transform (IST). Recent studies have shown that this characterization holds not only for initial value problems (IVPs), but also for initial-boundary value problems (IBVPs). The purpose of this talk is to characterize the solutions of IBVPs for discrete and continuous nonlinear Schrödinger (NLS) systems on a semi-infinite domain with linearizable boundary conditions (BCs).
The IBVP for the NLS equation $iq_t + q_{xx} + 2|q|^2 q = 0$ on $0 < x < \infty$ with homogeneous Dirichlet or Neumann BCs at the origin was studied in [1] using the IST on the whole line and an odd or even extension of the potential, respectively. The case of homogeneous Robin BCs, $q_x(0,t) - \alpha q(0,t) = 0$ with α a real constant, was also linearized in [2–4]. Importantly, in all of these methods the relation between solitons and discrete eigenvalues existing in the IVP is preserved in the IBVP. This leads to a paradox, however, since the soliton solutions of the NLS equation do not satisfy the linearizable BCs. A further paradox is that numerical solutions of the IBVP for the NLS equation show inequivocally that solitons are reflected at the boundary. But the soliton velocity is the real part of the discrete eigenvalue, which does not change in time. In this talk we show that the resolution of these apparent paradoxes is that discrete eigenvalues in the IBVP appear in *quartets*, as opposed to pairs in the IVP. This means that, for each soliton in the physical domain a symmetric counterpart exists, with equal amplitude and opposite velocity, whose presence ensures that the whole solution satisfies the BCs. The ostensible reflection of the soliton at the boundary of the physical domain then corresponds simply to the interchanging of roles between the "physical" and "mirror" solitons.
These results are similar in spirit to — and represents a nonlinear analogue of — the method of images that is used to solve boundary value problems in electrostatics. Here, however, the soliton reflection comes accompanied by a corresponding position shift, which is a reminder of the nonlinear nature of the problem. Similar phenomena arise for the focusing [5] and defocusing [6] NLS equations and also for the integrable discrete analogue of the NLS equation, the Ablowitz-Ladik (AL) lattice [7].

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